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Characterizations of Besov–Hardy–Sobolev Spaces via Harmonic Functions, Temperatures, and Related Means

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1. INTRODUCTION

If $s > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, then there exists a large number of equivalent characterizations of the (nowadays classical) Besov (or Lipschitz) spaces $B^s_{p,q}$, defined on the Euclidean n-space R_n. One can take one of these characterizations as a definition; e.g., $B^s_{p,q}$ (under the above restrictions for s, p and q) is the collection of all complex-valued functions $f(x) \in L_p$ (the usual L_p-spaces on R_n) such that

$$\|f|L_p\| + \left(\int_{R_n} |h|^{-sq} \|(\Delta^l_h f)(\cdot)|L_p\|^q \frac{dh}{|h|^n}\right)^{1/q} \tag{1}$$

is finite. We use standard notation: $\|f|L_p\|$ is the norm in L_p, and

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \qquad \Delta^k_h = \Delta^1_h(\Delta^{k-1}_h), \qquad k = 2, 3,..., \tag{2}$$

are the usual differences. In (1), l must be large enough, i.e., $l > s$. If $q = \infty$ then one has to modify (1) in the usual way. (We remind the reader that one can replace some differences in (1) by derivatives.) Under the above restrictions for s, p, q and l one can take (1) as a norm in $B^s_{p,q}$. Other equivalent norms in $B^s_{p,q}$ can be obtained with the help of the Gauß–Weierstraß semi-group (temperatures) or the Cauchy–Poisson semi-group (harmonic functions). If $t > 0$, then the Gauß–Weierstraß semi-group is given by

$$[W(t)f](x) = (4\pi t)^{-n/2} \int_{R_n} e^{-|x-y|^2/4t} f(y)\, dy, \tag{3}$$

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and the Cauchy–Poisson semi-group by

$$[P(t)f](x) = c_n \int_{R_n} \frac{t}{(|x-y|^2+t^2)^{(n+1)/2}} f(y)\, dy, \tag{4}$$

where c_n is an appropriate positive number and $f(x) \in L_p(R_n)$. Again let $s > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$. Then

$$\|f \,|\, L_p\| + \left(\int_0^\infty t^{(m-s/2)q} \left\| \frac{\partial^m W(t) f}{\partial t^m}(\cdot) \,|\, L_p \right\|^q \frac{dt}{t} \right)^{1/q} \tag{5}$$

with $m > s/2$ as well as

$$\|f \,|\, L_p\| + \left(\int_0^\infty t^{(m-s)q} \left\| \frac{\partial^m P(t) f}{\partial t^m}(\cdot) \,|\, L_p \right\|^q \frac{dt}{t} \right)^{1/q} \tag{6}$$

with $m > s$ are equivalent norms in $B^s_{p,q}$. (In both cases, m is an integer. Furthermore, if $q = \infty$, then (5) and (6) must be modified in the usual way.) The restriction $s > 0$ can be removed as follows. If $-\infty < s < \infty$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$, and $\kappa > s/2$, then

$$\left(\int_0^\infty t^{(\kappa - s/2)q} \|F^{-1}[(1+|\xi|^2)^\kappa e^{-t|\xi|^2} Ff](\cdot) \,|\, L_p\|^q \frac{dt}{t} \right)^{1/q} \tag{7}$$

is an equivalent norm in $B^s_{p,q}$ (modification if $q = \infty$). We used standard notations: F and F^{-1} are the Fourier transform and its inverse, respectively, in S' (the collection of all complex-valued tempered distributions on R_n). It is easy to see that (7) is the generalization of (5) if one takes into consideration that $W(t)f$ satisfies the heat equation in the half-space $\{(x,t) \,|\, x \in R_n, t > 0\}$ and that $F(e^{-|\xi|^2/2})(x) = e^{-|x|^2/2}$. In a similar way one can extend (6) to $B^s_{p,q}$ with $-\infty < s < \infty$, $1 < p < \infty$, $1 \leqslant q \leqslant \infty$. (Then one has to replace $e^{-t|\xi|^2}$ in (7) by $e^{-t|\xi|}$.) The classical paper about this material is [14] (cf. also [1, Chap. 4]). The above version can be found in [17, pp. 190–196]. In recent years the spaces $B^s_{p,q}$ (always defined on R_n) have been extended to $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. There are two versions; nonhomogeneous spaces, which cover the above classical spaces, and homogeneous spaces, which will be denoted in the sequel by $\dot{B}^s_{p,q}$. Besides these spaces we consider a second scale $F^s_{p,q}$ (resp. $\dot{F}^s_{p,q}$), where $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqslant \infty$. Special cases of that scale are the classical Bessel-potential (or Lebesgue or Liouville) spaces, which contain the Sobolev spaces, and the Hardy spaces. The aim of this paper is to find characterizations of all these spaces in the spirit of (5)–(7). However, our approach has nothing to do with semi-groups. Roughly speaking, we replace $t^\kappa(1+|\xi|^2)^\kappa e^{-t|\xi|^2}$ in (7) by $\varphi(tx)$ with $t > 0$ and $x \in R_n$ and ask for conditions upon the function $\varphi(x)$ such that the counterpart of (7) yields an equivalent norm (more precisely: quasi-norm) on $\dot{B}^s_{p,q}$. A similar question is asked for $\dot{F}^s_{p,q}$ (and also for the non-homogeneous counterparts $B^s_{p,q}$ and $F^s_{p,q}$). But mostly we shall be concerned with the homogeneous spaces.

The paper is organized as follows. Section 2 contains the necessary definitions and also some remarks about the above-mentioned special spaces. The results are formulated in Section 3: Theorem 1 and Theorem 2 in 3.1 (homogeneous spaces), Theorem 3 in 3.2 (non-homogeneous spaces), and a discussion of the above special cases (Gauß–Weierstraß semi-group and Cauchy–Poisson semi-group) in 3.3. Proofs of the theorems are given in Section 4. As usual, c and c' denote general positive constants, which may differ from line to line.

2. Definitions

2.1. *The Homogeneous Spaces*

Let R_n be the n-dimensional real Euclidean space. Let S be the Schwartz space of all complex-valued infinitely differentiable rapidly decreasing functions on R_n, and let S' be the collection of all tempered distributions on R_n (the dual of S). Let $\rho(x) \in S$ be a non-negative function with

$$\operatorname{supp} \rho \subset \{y \mid \tfrac{1}{2} < |y| < 2\} \tag{8}$$

and

$$\sum_{j=-\infty}^{\infty} \rho(2^{-j}x) = 1 \qquad \text{if} \quad 0 \neq x \in R_n \tag{9}$$

(there exist functions ρ with the required properties). F denotes the Fourier transform on S' and F^{-1} the inverse Fourier transform. If $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ then

$$\dot{B}^s_{p,q} = \Big\{ f \mid f \in S', \|f \mid \dot{B}^s_{p,q}\| = \Big(\sum_{j=-\infty}^{\infty} 2^{jsq} \|F^{-1}\rho(2^{-j}\,\cdot)Ff \mid L_p\|^q \Big)^{1/q} < \infty \Big\}. \tag{10}$$

We recall that

$$\|h \mid L_p\| = \Big(\int_{R_n} |h(x)|^p \, dx \Big)^{1/p} \qquad \text{if} \quad 0 < p < \infty$$

and

$$\|h \,|\, L_\infty\| = \operatorname*{ess\,sup}_{x \in R_n} |f(x)|.$$

Further, $F^{-1}\rho(2^{-j}\,\cdot)\,Ff$ means that F^{-1} is applied to $\rho(2^{-j}x)\,Ff$. If $q = \infty$, then one has to replace $(\sum_{j=-\infty}^{\infty} a_j^q)^{1/q}$ in (10) by $\sup_j a_j$. If $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqslant \infty$, then

$$\dot{F}^s_{p,q} = \Big\{ f \,|\, f \in S',\ \|f \,|\, \dot{F}^s_{p,q}\| = \Big\| \Big(\sum_{j=-\infty}^{\infty} 2^{jsq} \,|(F^{-1}\rho(2^{-j}\,\cdot)\,Ff)(\cdot)|^q \Big)^{1/q} \Big| L_p \Big\| < \infty \Big\} \qquad (11)$$

(with the above modification if $q = \infty$). The spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ should be considered modulo polynomials. If $P(x)$ is a polynomial in R_n, then

$$\|f \,|\, \dot{B}^s_{p,q}\| = \|f + P \,|\, \dot{B}^s_{p,q}\|$$

(and similarly for $\dot{F}^s_{p,q}$). But we do not stress that point any more, because it is not of interest for our purpose in this paper. A more detailed discussion of this somewhat delicate question has been given in [21, 5.1] (cf. also [19, Chap. 3]).

2.2. *The Non-Homogeneous Spaces*

Let $\rho(x)$ be the function from 2.1, and let

$$\rho_0(x) = 1 - \sum_{j=1}^{\infty} \rho(2^{-j}x) \qquad \text{if} \quad x \in R_n.$$

If $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$, then

$$B^s_{p,q} = \Big\{ f \,|\, f \in S',\ \|f \,|\, B^s_{p,q}\| = \|F^{-1}\rho_0 Ff \,|\, L_p\| + \Big(\sum_{j=1}^{\infty} 2^{jsq} \,\|F^{-1}\rho(2^{-j}\,\cdot)\,Ff \,|\, L_p\|^q \Big)^{1/q} < \infty \Big\} \qquad (12)$$

(usual modification if $q = \infty$). If $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqslant \infty$, then

$$F^s_{p,q} = \Big\{ f \,|\, f \in S',\ \|f \,|\, F^s_{p,q}\| = \|F^{-1}\rho_0 Ff \,|\, L_p\| + \Big\| \Big(\sum_{j=1}^{\infty} 2^{jsq} \,|(F^{-1}\rho(2^{-j}\,\cdot)\,Ff)(\cdot)|^q \Big)^{1/q} \Big| L_p \Big\| < \infty \Big\} \qquad (13)$$

(usual modification if $q = \infty$).

2.3. *Remarks and Special Spaces*

The spaces from 2.1 and 2.2 have been studied extensively in [9, 19, 21]. The definition of the spaces $\dot{B}^s_{p,q}$ and $B^s_{p,q}$ in the above manner goes back to J. Peetre [6, 7]. The spaces $F^s_{p,q}$ with $p > 1$ and $q > 1$ have been introduced in [16]; the extension to $0 < p < \infty$ and $0 < q \leqslant \infty$ is again due to Peetre [8]. (Similar spaces restricted to $p > 1$ and $q > 1$ have also been introduced by P. I. Lizorkin [4, 5].) We do not describe properties of these spaces, but we mention that the spaces from (10)–(13) are independent of the function $\rho(x)$: Different ρ's yield equivalent quasi-norms. We describe some special cases:

(i) If $s > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$ then the spaces from (12) coincide with the classical Besov spaces from the Introduction.

(ii) If $s > 0$ then $B^s_{\infty,\infty} = \mathscr{C}^s$ are the classical Hölder–Zygmund spaces.

(iii) If $-\infty < s < \infty$ and $1 < p < \infty$, then $F^s_{p,2} = H^s_p$ are the usual Bessel-potential spaces; in particular, if $1 < p < \infty$ and $m = 0, 1, 2, \ldots$, then $F^m_{p,2} = W^m_p$ are the usual Sobolev spaces.

(iv) If $0 < p < \infty$ then $\dot{F}^0_{p,2} = H_p$ are the Hardy spaces in the sense of C. Fefferman and E. M. Stein [3].

Proofs of these assertions and more detailed references may be found in [19, 21].

As has been said, the aim of this paper is to describe characterizations of the spaces under consideration in the spirit of (5)–(7). For the classical Besov spaces $B^s_{p,q}$ with $s > 0$, $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$, this has been done by M. H. Taibleson [14]. The extension of these results to $B^s_{p,q}$ (and $\dot{B}^s_{p,q}$) with $p < 1$ faces some difficulties. In particular, the semi-group approach has no immediate counterpart. Characterizations of $B^s_{p,q}$ and $\dot{B}^s_{p,q}$ with $p < 1$ in the sense of (6) have been obtained by Peetre [9, p. 256] ($0 < p < 1$, $m = 1$, $s < 1$) and in the sense of (5) by Bui Huy Qui [10].

3. Results

3.1. *The Homogeneous Spaces*

Let $f \in S'$ and let $\varphi(x) \in S$. If $t > 0$ then we introduce the maximal function

$$(\varphi(t\,\cdot)f)^*(x) = \sup_{y \in R_n} (1 + |t^{-1}y|^a)^{-1} \, |(F^{-1}\varphi(t\,\cdot)Ff)(x-y)| \qquad (14)$$

with $a > 0$ (later on we choose a big enough). This makes sense at least if $\varphi(x)$ has a compact support (and if ∞ is an admissible value for the left-hand side of (14)). In the theorems below we use this maximal function for a bigger class of admissible functions φ (and under some restrictions for f). One can always understand the corresponding expressions as a result of a limiting process, where one starts with smooth φ's with compact support in R_n (or even in $R_n - \{0\}$). In that sense the maximal function from (14) and also $F^{-1}\varphi(t\,\cdot)\,Ff$ make sense for all involved φ's and f's. But we shall not stress this point in the sequel.

THEOREM 1. *Let L be a natural number and let $\varphi(x)$ be a non-negative infinitely differentiable function in $R_n - \{0\}$ such that*

$$\sup_{x \in R_n - \{0\}} (|x|^L + |x|^{-L})\,|D^\alpha \varphi(x)| < \infty \qquad \text{if} \quad |\alpha| \leqslant L \tag{15}$$

and $\varphi(x) > 0$ if $\frac{1}{4} < |x| < 4$.

(i) *Let $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. If $a > n/p$ in* (14) *and if L in* (15) *is big enough then*

$$\left(\int_0^\infty t^{-sq}\,\|F^{-1}\varphi(t\,\cdot)\,Ff\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q} \tag{16}$$

and

$$\left(\int_0^\infty t^{-sq}\|(\varphi(t\,\cdot)f)^*\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q} \tag{17}$$

are equivalent quasi-norms in $\dot{B}^s_{p,q}$ (modification if $q = \infty$).

(ii) *Let $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqslant \infty$. If $a > n/\min(p, q)$ in* (14) *and if L in* (15) *is big enough then*

$$\left\|\left(\int_0^\infty t^{-sq}\,|(F^{-1}\varphi(t\,\cdot)\,Ff)(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\| \tag{18}$$

and

$$\left\|\left(\int_0^\infty t^{-sq}\,|(\varphi(t\,\cdot)f)^*\,(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|\,L_p\right\| \tag{19}$$

are equivalent quasi-norms in $\dot{F}^s_{p,q}$ (modification if $q = \infty$).

Remark 1. "L big enough" means

$$L \geqslant |s| + 6n/p + n + 4 \tag{20}$$

in part (i) and

$$L \geqslant |s| + 6n/\min(p, q) + n + 4 \tag{21}$$

in part (ii) of the theorem, provided that one chooses a in (14) near to n/p and $n/\min(p, q)$, respectively. However, the numbers on the right-hand side of (20) and (21) are somewhat artificial. The search for best numbers seems to be a hard task, at least for the spaces $\dot{F}^s_{p,q}$. On the other hand, one can find rather natural conditions for $\varphi(x)$ under which (16) is an equivalent quasi-norm in $\dot{B}^s_{p,q}$.

THEOREM 2. *Let $\varphi(x)$ be a non-negative infinitely differentiable function on $R_n - \{0\}$ such that $\varphi(x) > 0$ if $\frac{1}{4} < |x| < 4$. Let $-\infty < s < \infty$, $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$. Let*

$$\begin{aligned} \sigma_p &= n/2 && \text{if} \quad 1 \leqslant p \leqslant \infty \\ \sigma_p &= n(1/p - \tfrac{1}{2}) && \text{if} \quad 0 < p \leqslant 1 \end{aligned} \qquad \text{and} \quad \tilde{\sigma}_p = \sigma_p - n/2. \tag{22}$$

Let B be the unit ball in R_n and let $s_0 + \tilde{\sigma}_p < s < s_1$. If

$$|x|^{-s_1} \varphi(x) \in H^\lambda_2(B) \quad \text{with} \quad \lambda > \sigma_p, \tag{23}$$

and

$$\sup_{|x| \geqslant 1} |x|^{|\alpha|} \, |D^\alpha(|x|^{-s_0} \varphi(x))| < \infty \qquad \text{for } |\alpha| \leqslant [\sigma_p] + 1, \tag{24}$$

then (16) is an equivalent quasi-norm in $\dot{B}^s_{p,q}$.

Remark 2. We recall that $H^\lambda_2 = H^\lambda_2(R_n)$ is the usual Bessel-potential space on R_n and that $H^\lambda_2(B)$ is the restriction of H^λ_2 to the unit ball B. If $\lambda > 0$ is an integer then H^λ_2 and $H^\lambda_2(B)$ are the usual Sobolev spaces on R_n and on B, respectively. Furthermore, $[\sigma_p]$ is the biggest integer less than or equal to σ_p. In particular, the theorem shows that the behavior of $\varphi(x)$ near 0 and at infinity is quite different if $p < 1$. Conditions (23) and (24) are fairly natural. One can reformulate (24) also in the language of the spaces H^β_2.

3.2. *The Non-Homogeneous Spaces*

In order to find counterparts of Theorems 1 and 2 from 3.1 for the non-homogeneous spaces, it is desirable (but not absolutely necessary) to include a term of the type $\|f|L_p\|$. If $1 \leqslant p \leqslant \infty$ this means that we must restrict s in $B^s_{p,q}$ and $F^s_{p,q}$ by $s > 0$. The case where $0 < p < 1$, a bit more complicated question, is discussed in [21, Remark 2.5.3/1]. It turns out that $f \in L_p$ makes sense if $f \in B^s_{p,q}$ or $f \in F^s_{p,q}$ and $s > n(1/p - 1)$. However, for technical reasons we sometimes need stronger restrictions.

THEOREM 3. (i) *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$ *and* $s>n/p$. *Under the same conditions for* $\varphi(x)$, L, *and as in Theorem* 1(i),

$$\|f\,|L_p\| + \left(\int_0^\infty t^{-sq}\,\|F^{-1}\varphi(t\,\cdot)\,Ff\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q} \tag{25}$$

and

$$\|f\,|L_p\| + \left(\int_0^\infty t^{-sq}\,\|(\varphi(t\,\cdot)f)^*\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q}. \tag{26}$$

are equivalent quasi-noms in $B^s_{p,q}$ *(modification if* $q=\infty$*).*

(ii) *Let* $0<p<\infty$, $0<q\leqslant\infty$ *and* $s>n/\min(p,q)$. *Under the same conditions for* $\varphi(x)$, L, *and* a *as in Theorem* 1(ii),

$$\|f\,|L_p\| + \left\|\left(\int_0^\infty t^{-sq}\,|(F^{-1}\varphi(t\,\cdot)\,Ff)(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\| \tag{27}$$

and

$$\|f\,|L_p\| + \left\|\left(\int_0^\infty t^{-sq}\,|(\varphi(t\,\cdot)f)^*\,(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\| \tag{28}$$

are equivalent quasi-norms in $F^s_{p,q}$ *(modification if* $q=\infty$*).*

(iii) *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$ *and* $s>\max(0, n(1/p-1))$. *Under the same conditions for* $\varphi(x)$ *as in Theorem* 2, *formula* (25) *gives an equivalent quasi-norm in* $B^s_{p,q}$.

3.3. *Special Functions* $\varphi(x)$, *Remarks*

One can identify the function $\varphi(x)$ from 3.1 with the function $\rho(x)$ from 2.1 (after replacing 2^{-1} and 2 in (8) by 4^{-1} and 4, respectively). Then (16) and (18) are essentially the continuous versions of $\|f\,|\dot{B}^s_{p,q}\|$ and $\|f\,|\dot{F}^s_{p,q}\|$ from (10) and (11), respectively. More interesting are functions $\varphi(x)$ which have no compact support. If m is a sufficiently large number, then

$$\varphi(x)=|x|^{2m}\,e^{-|x|^2}, \qquad x\in R_n, \tag{29}$$

satisfies the hypotheses of Theorem 1. On the other hand it is well known that

$$\begin{aligned}(F^{-1}\varphi(\sqrt{t}\,\cdot)\,Ff)(x) &= t^m(-\Delta)^m\,(F^{-1}e^{-t|\xi|^2}Ff)(x)\\ &= ct^m\left[\frac{\partial^m}{\partial t^m}\,W(t)f\right](x), \qquad c\neq 0,\end{aligned} \tag{30}$$

where $W(t)f$ is the Gauß–Weierstraß semi-group (cf. (3) and [17, p. 191/192]).

COROLLARY 1. (i) *Let* $-\infty < s < \infty$, $0 < p \leqslant \infty$ *and* $0 < q \leqslant \infty$. *If* m *is a non-negative integer with* $2m > s$ *then*

$$\left(\int_0^\infty t^{(m-s/2)q} \left\| \frac{\partial^m W(t)f}{\partial t^m}(\cdot)|L_p \right\|^q \frac{dt}{t}\right)^{1/q} \tag{31}$$

is an equivalent quasi-norm in $\dot{B}^s_{p,q}$ *(modification if* $q = \infty$*).*

(ii) *Let* $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q \leqslant \infty$. *If* m *is a sufficiently large natural number then*

$$\left\| \left(\int_0^\infty t^{(m-s/2)q} \left| \frac{\partial^m W(t)f}{\partial t^m}(\cdot)\right|^q \frac{dt}{t}\right)^{1/q} |L_p \right\| \tag{32}$$

is an equivalent quasi-norm in $\dot{F}^s_{p,q}$ *(modification if* $q = \infty$*).*

Proof. The proof is an easy consequence of (29) and (30) on the one hand and Theorems 1(ii) and 2 on the other hand ($s_1 = 2m$ in (23) in part (i)).

Remark 3. This is a representation of $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ via temperatures.

Another interesting example of an admissible function $\varphi(x)$ is given by

$$\varphi(x) = |x|^m e^{-|x|}, \qquad x \in R_n. \tag{33}$$

If m is a sufficiently large natural number, then the hypotheses of Theorem 1 are satisfied. If $t > 0$ then we have

$$\begin{aligned}(F^{-1}\varphi(t\,\cdot)Ff)(x) &= t^m(F^{-1}|\xi|^m e^{-t|\xi|}Ff)(x)\\ &= (-t)^m \frac{\partial^m}{\partial t^m}(F^{-1}e^{-t|\xi|}Ff)(x)\\ &= ct^m\left[\frac{\partial^m}{\partial t^m}P(t)f\right](x),\end{aligned} \tag{34}$$

where $P(t)f$ is the Cauchy–Poisson semi-group (cf. (4) and $c \neq 0$). The last formula in (34) may be found in [17, p. 195].

COROLLARY 2. (i) *Let* $-\infty < s < \infty$, $0 < p \leqslant \infty$ *and* $0 < q \leqslant \infty$. *If* m *is a nonnegative integer with*

$$\begin{aligned} &m > s && \text{if}\quad 1 \leqslant p \leqslant \infty,\\ &m > s + n(1/p - 1) && \text{if}\quad 0 < p < 1, \end{aligned} \tag{35}$$

then

$$\left(\int_0^\infty t^{(m-s)q} \left\| \frac{\partial^m P(t) f}{\partial t^m}(\cdot) | L_p \right\|^q \frac{dt}{t}\right)^{1/q} \tag{36}$$

is an equivalent quasi-norm in $\dot{B}^s_{p,q}$ *(modification if* $q = \infty$*).*

(ii) *Let* $-\infty < s < \infty$, $0 < p < \infty$ *and* $0 < q \leqslant \infty$. *If* m *is a sufficiently large natural number then*

$$\left\| \left(\int_0^\infty t^{(m-s)q} \left| \frac{\partial^m P(t) f}{\partial t^m}(\cdot) \right|^q \frac{dt}{t}\right)^{1/q} | L_p \right\| \tag{37}$$

is an equivalent quasi-norm in $\dot{F}^s_{p,q}$ *(modification if* $q = \infty$*).*

Proof. Part (ii) is again an immediate consequence of Theorem 1(ii), (33) and (34). The proof of part (i) is a bit more complicated than the proof of Corollary 1(*i*), because $\varphi(x)$ from (33) is not smooth at the origin (in contrast to the function from (29)). If $\varphi(x)$ is given by (33) and (35), then we have to prove that (23) is satisfied. Let $s_1 > s$. Then it follows that near the origin

$$D^\alpha(|x|^{m-s_1} e^{-|x|}) = \mathcal{O}(|x|^{m-s_1-|\alpha|}). \tag{38}$$

If λ is a non-negative integer then we have

$$|x|^{m-s_1} e^{-|x|} \in H_2^\lambda(B) \qquad \text{if} \quad m > \lambda - n/2 + s_1. \tag{39}$$

By complex interpolation it follows that (39) is true for all $\lambda \geqslant 0$. Because $\lambda > \sigma_p$, this coincides with (35).

Remark 4. In contrast to $2m > s$ in Corollary 1(i), the number m in (35) depends on p if $p < 1$. One can improve (35) slightly, but not essentially. If, e.g., $s < 0$ and $s_1 = m = 0$, then $D^\alpha e^{-|x|} = \mathcal{O}(|x|^{-|\alpha|+1})$ if $|\alpha| \geqslant 1$ near the origin. By the above interpolation argument we have $e^{-|x|} \in H_2^\lambda(R_n)$ if $-\lambda + 1 > -n/2$, i.e., $\lambda < n/2 + 1$. In other words, $\lambda > \sigma_p$ from (23) can be satisfied if either $\infty \geqslant p \geqslant 1$ or $0 < p \leqslant 1$ and $n/p < n + 1$. This shows that (36), with $s < 0$, $0 < q \leqslant \infty$ and $m = 0$, is an equivalent quasi-norm in $\dot{B}^s_{p,q}$ if $\infty \geqslant p > n/(n+1)$.

Remark 5. One can try to study other special functions $\varphi(x)$ in the sense of Theorems 1–3. Let

$$\varphi(x) = |x|^{-m} (e^{ixh} - 1)^M, \tag{40}$$

where xh is the scalar product of $x \in R_n$ and $h \in R_n$. If one chooses m and M in an appropriate way, then (15) with a given L is satisfied (but not the other assumptions for $\varphi(x)$). What makes (40) (and modifications of (40)) interesting is the fact that

$$(F^{-1}\varphi(\cdot)\,Ff)(x) = \Delta_h^M(F^{-1}\,|\xi|^{-m}\,Ff)(x),$$

where Δ_h^M are the usual differences in R_n. One can follow that path, but not as an immediate application of the above theorems. However, it is possible to obtain characterizations of the spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ via derivatives and differences in that way. For details we refer to [21, 2.5.9–2.5.12].

Remark 6. We recall that $H_p = \dot{F}^0_{p,2}$ if $0 < p < \infty$, where H_p are the n-dimensional Hardy spaces. Now one can compare the above characterization of H_p via (18) and the characterization given by Fefferman and Stein in [3]. Let $\varphi(x)$ be a function which satisfies the hypotheses of Theorem 1(ii) (with $s=0$ and $q=2$) and let $\psi(x) \in S$ with $\psi(0)=1$, then

$$\left\| \left(\int_0^\infty |(F^{-1}\varphi(t\,\cdot)\,Ff)(\cdot)|^2\,dx \right)^{1/2} |L_p \right\| \tag{41}$$

and

$$\|\sup_{t>0} |(F^{-1}\psi(t\,\cdot)\,Ff)(\cdot)|\,|L_p\| \tag{42}$$

are equivalently quasi-norms for H_p with $0<p<\infty$. Here, (42) comes from [3] and (41) is the specialization of (18).

Remark 7. Finally we mention an application of the two corollaries. Recently F. Ricci and M. H. Taibleson obtained in [11] representation theorems for harmonic functions in $R_2^+ = \{(x,t)\,|\,x \in R_1, t>0\}$ via atoms and molecules (cf. also [12, 15]). This approach has been extended to temperatures in R_2^+ by S. E. Sands (cf. [13]). On the basis of the above corollaries one obtains corresponding representations by atoms and molecules for the spaces $\dot{B}^s_{p,q}$. This possibility has been pointed out explicitly in [13] (and it was the starting point for the present paper).

Finally we formulate the non-homogeneous counterparts of Corollaries 1 and 2.

COROLLARY 3. (i) *Let* $0<p\leqslant\infty$, $0<q\leqslant\infty$ *and* $s>\max(0, n(1/p-1))$. *If* m *and* k *are natural numbers with* $2m>s$ *and*

$$k>s \qquad \textit{if} \quad 1\leqslant p\leqslant\infty,$$

$$k>s+n(1/p-1) \qquad \textit{if} \quad 0<p<1,$$

then

$$\|f|L_p\| + \left(\int_0^\infty t^{(m-s/2)q} \left\|\frac{\partial^m W(t)f}{\partial t^m}(\cdot)|L_p\right\|^q \frac{dt}{t}\right)^{1/q} \tag{43}$$

and

$$\|f|L_p\| + \left(\int_0^\infty t^{(k-s)q} \left\|\frac{\partial^k P(t)f}{\partial t^k}(\cdot)|L_p\right\|^q \frac{dt}{t}\right)^{1/q} \tag{44}$$

are equivalent quasi-norms in $B^s_{p,q}$ *(modification if* $q=\infty$*).*

(ii) *Let* $0<p<\infty$, $0<q\leqslant\infty$ *and* $s>n/\min(p,q)$. *If* m *is a sufficiently large natural number then*

$$\|f|L_p\| + \left\|\left(\int_0^\infty t^{(m-s/2)q} \left|\frac{\partial^m W(t)f}{\partial t^m}(\cdot)\right|^q \frac{dt}{t}\right)^{1/q}|L_p\right\| \tag{45}$$

and

$$\|f|L_p\| + \left\|\left(\int_0^\infty t^{(m-s)q} \left|\frac{\partial^m P(t)f}{\partial t^m}(\cdot)\right|^q \frac{dt}{t}\right)^{1/q}|L_p\right\| \tag{46}$$

are equivalent quasi-norms in $F^s_{p,q}$ *(modification if* $q=\infty$*).*

Proof. The proof is the same as the proofs of Corollaries 1 and 2 if one uses Theorem 3 instead of Theorems 1 and 2.

Remark 8. If $1<p<\infty$ and $1\leqslant q\leqslant\infty$ then (43) and (44) coincide with (5) and (6), respectively.

4. Proofs

4.1. *Proof of Theorem* 1

We prove part (ii). The proof of part (i) is the same, but some details are simpler (cf. also Step 3).

Step 1. We recall that we always assume that $\varphi(x)$ is sufficiently smooth. Then we have, under the hypotheses of Theorem 1(ii), the maximal inequality

$$\left\|\left(\sum_{j=-\infty}^{\infty} 2^{-jsq} \sup_{1\leqslant\lambda\leqslant 2} |(\varphi(2^j\lambda\,\cdot)f)^*(\cdot)|^q\right)^{1/q}|L_p\right\| \leqslant c\,\|f|\dot{F}^s_{p,q}\|, \tag{47}$$

where c is independent of $f\in\dot{F}^s_{p,q}$. A proof of the non-homogeneous version of this estimate may be found in [21, 2.3.6] (cf. also [19, p. 29]). There is no difficulty in extending this estimate to the above homogeneous version. However, it is easy to see that the quasi-norm in (19) can be estimated from above by the left-hand side of (47) and hence by $c\,\|f|\dot{F}^s_{p,q}\|$. Because

$$|(F^{-1}\varphi(t\,\cdot)\,Ff)(x)| \leqslant (\varphi(t\,\cdot)f)^*\,(x), \qquad x \in R_n, \tag{48}$$

it follows that the quasi-norms in (18) and (19) can be estimated from above by $c\,\|f|\dot{F}^s_{p,q}\|$.

Step 2. We prove that $\|f|\dot{F}^s_{p,q}\|$ can be estimated from above by the quasi-norm in (18). Afterwards it follows from (48) and Step 1 that (18) and (19) are equivalent quasi-norms in $\dot{F}^s_{p,q}$. Let $\rho(x)$ be the function from 2.1 and let $1 \leqslant \lambda \leqslant 2$. Then we have

$$\begin{aligned}
&(F^{-1}\rho(2^{-j}\,\cdot)\,Ff)(x)\\
&= \left(F^{-1}\frac{\rho(2^{-j}\,\cdot)}{\varphi(2^{-j}\lambda\,\cdot)}\,\varphi(2^{-j}\lambda\,\cdot)\,Ff\right)(x)\\
&= \int_{R_n}\left(F^{-1}\frac{\rho(2^{-j}\,\cdot)}{\varphi(2^{-j}\lambda\,\cdot)}\right)(y)(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(x-y)\,dy.
\end{aligned} \tag{49}$$

Under the assumptions for the supports of ρ and φ, the first factor in the integral in (49) is a smooth function, which can be calculated by $(F^{-1}(\rho/\varphi(\lambda\,\cdot)))(2^jy)\,2^{jn}$ and estimated from above by $c2^{jn}(1+|2^jy|)^{-b}$, where $b>0$ is at our disposal. Let $0<r<\min(p,q,1)$. Then we obtain, with the help of (14), that

$$\begin{aligned}
&|(F^{-1}\rho(2^{-j}\,\cdot)\,Ff)(x)|\\
&\quad\leqslant c(\varphi(2^{-j}\lambda\,\cdot)f)^{*\,1-r}\,(x)\\
&\quad\times\int_{R_n} 2^{jn}\,\frac{(1+|2^jy|^a)^{1-r}}{(1+|2^jy|)^b}\,|(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(x-y)|^r\,dy.
\end{aligned} \tag{50}$$

In the first factor on the right-hand side of (50) we take the supremum with respect to λ, where $1 \leqslant \lambda \leqslant 2$. Afterwards we integrate the modified inequality with respect to λ (which appears now only in the integral). Because $r<q$ we have

$$\begin{aligned}
&\int_1^2 |(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(x-y)|^r\,d\lambda\\
&\qquad\leqslant \left(\int_1^2 |(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(x-y)|^q\,d\lambda\right)^{r/q}.
\end{aligned}$$

Finally we replace the integration over R_n in (50) by integrations over $B_{-j}=$

$\{y \mid |y| \leqslant 2^{-j}\}$ and $\{y \mid 2^{-j+l} \leqslant |y| \leqslant 2^{-j+l+1}\}$ with $l=0, 1, 2,\ldots$. Then we have

$$|(F^{-1}\rho(2^{-j}\cdot)\, Ff)(x)|$$
$$\leqslant c \sup_{1\leqslant\mu\leqslant 2} (\varphi(2^{-j}\mu\cdot f)^{*1-r}(x)$$
$$\times \sum_{l=0}^{\infty} 2^{jn}2^{-ld} \int_{B_{-j+l}} \left(\int_1^2 |(F^{-1}\varphi(2^{-j}\lambda\cdot)\, Ff)(x-y)|^q \, d\lambda\right)^{r/q} dy, \tag{51}$$

where $d>0$ is at our disposal (a and r in (50) are fixed). Let Mh be the Hardy–Littlewood maximal function of a given function h. If we choose $d>n$ then every term of $\sum_{l=0}^{\infty}\cdots$ in (51) can be estimated from above by

$$2^{-l(d-n)}\left[M\left(\int_1^2 |F^{-1}\varphi(2^{-j}\lambda\cdot)\, Ff|^q \, d\lambda\right)^{r/q}\right](x).$$

Consequently,

$$|(F^{-1}\rho(2^{-j}\cdot)\, Ff)(x)| \leqslant c \sup_{1\leqslant\mu\leqslant 2} (\varphi(2^{-j}\mu\cdot)f)^{*1-r}(x)$$
$$\times\left[M\left(\int_1^2 |(F^{-1}\varphi(2^{-j}\lambda\cdot)\, Ff)(\cdot)|^q \, d\lambda\right)^{r/q}\right](x). \tag{52}$$

We multiply both sides with 2^{sj} and apply the l_q-quasi-norm. By Hölder's inequality, based on $1/q=(1-r)/q-r/q$, and obvious abbreviations it follows that

$$\|2^{js}(F^{-1}\rho(2^{-j}\cdot)\, Ff)(x)\,|\,l_q\|$$
$$\leqslant c\,\|2^{js}\sup_{1\leqslant\mu\leqslant 2}(\varphi(2^{-j}\mu\cdot)f)^*(x)\,|\,l_q\|^{1-r}\,\|2^{jsr}M(\cdots)^{r/q}(x)\,|\,l_{q/r}\|. \tag{53}$$

We take the L_p-quasi-norm with respect to x. Again by Hölder's inequality, based on $1/p=(1-r)/p+r/p$, we obtain that

$$\|2^{js}(F^{-1}\rho(2^{-j}\cdot)\, Ff)(x)\,|\,l_q|\,L_p\|$$
$$\leqslant c\,\|2^{js}\sup_{1\leqslant\mu\leqslant 2}(\varphi(2^{-j}\mu\cdot)f)^*(\cdot)\,|\,l_q|\,L_p\|^{1-r}$$
$$\times\|2^{jsr}M(\cdots)^{r/q}(\cdot)\,|\,l_{q/r}|\,L_{p/r}\|. \tag{54}$$

The left-hand side of (54) coincides with $\|f\,|\,\dot{F}^s_{p,q}\|$. The first factor on the right-hand side of (54) can be estimated from above by $c\,\|f\,|\,\dot{F}^s_{p,q}\|^{1-r}$ (cf. (47)). We assume that $q<\infty$. Then $\infty>q/r>1$ and $\infty>p/r>1$. This shows that we can apply the vector-valued Hardy–Littlewood maximal inequality due to Fefferman and Stein [2] to the second factor on the right-hand side of (54). Then this factor can be estimated from above by

$$\begin{aligned} &c\,\|2^{jsr}(\cdots)^{r/q}(\cdot)\,|l_{q/r}|\,L_{p/r}\| \\ &= c\,\left\|\left(\sum_{j=-\infty}^{\infty}\int_1^2 |2^{js}(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(\cdot)|^q\,d\lambda\right)^{r/q}|L_{p/r}\right\| \\ &\leqslant c'\,\left\|\left(\int_0^\infty |t^{-s}(F^{-1}\varphi(t\,\cdot)\,Ff)(\cdot)|^q\,\frac{dt}{t}\right)^{r/q}|L_{p/r}\right\| \\ &= c'\,\left\|\left(\int_0^\infty t^{-sq}\,|(F^{-1}\varphi(t\,\cdot)\,Ff)(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\|^r. \end{aligned} \tag{55}$$

If we put together these estimates then it follows from (54) that $\|f\,|\dot{F}^s_{p,q}\|$ can be estimated from above by the quasi-norm in (18). This completes the proof provided that $q < \infty$.

Step 3. If $q = \infty$ then the second factor in (54) must be replaced by

$$\|M[\sup_{j,\lambda} 2^{js}\,|(F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff)(\cdot)|]^r\,|L_{p/r}\|.$$

Because $\infty > p/r > 1$, we can apply the scalar version of the Hardy–Littlewood maximal inequality, which yields the desired result. The proof is complete. In the case of the spaces $\dot{B}^s_{p,q}$, the scalar version of the Hardy–Littlewood maximal inequality is sufficient. But this inequality is also valid if $p = \infty$. This makes clear that $p = \infty$ is an admissible value in part (i) of Theorem 1.

4.2. *Proof of Theorem* 2

Step 1. We prove that there exists a constant c such that (under the hypotheses of Theorem 2)

$$\left(\int_0^\infty t^{-sq}\,\|F^{-1}\varphi(t\,\cdot)\,Ff\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q} \leqslant c\,\|f\,|\dot{B}^s_{p,q}\| \tag{56}$$

holds for every $f \in \dot{B}^s_{p,q}$. The function $\rho(x)$ always has the meaning of 2.1, in particular,

$$f = \sum_{j=-\infty}^{\infty} F^{-1}\rho(2^{-j}\,\cdot)\,Ff \tag{57}$$

(we always assume, without restriction of generality, that Ff vanishes near the origin). The left-hand side of (56) can be estimated from above by

$$c\left(\sum_{l=-\infty}^{\infty} 2^{lsq}\int_{2^{-l}}^{2^{-l+1}} \|F^{-1}\varphi(t\,\cdot)\,Ff\,|L_p\|^q\,dt\,2^l\right)^{1/q}. \tag{58}$$

Let $2^{-l} \leqslant t \leqslant 2^{-l+1}$. We put (57) in (58) and split $\sum_{j=-\infty}^{\infty}$ in the lth term, in $\sum_{j=-\infty}^{l}$ and $\sum_{j=l+1}^{\infty}$. First we deal with the terms with $j \leqslant l$. Let $h(x) \in S$ be a function with $h(x) = 1$ if $|x| \leqslant 2$ and $h(x) = 0$ if $|x| \geqslant 4$. If s_1 has the meaning of the theorem, i.e., $s_1 > s$, then $\tilde{\rho}(x) = |x|^{s_1} \rho(x)$ has essentially the same properties as $\rho(x)$ (in particular, (10) with $\tilde{\rho}$ instead of ρ is a characterization of $\dot{B}^s_{p,q}$ in the sense of equivalent quasi-norms). Let $1 \leqslant p \leqslant \infty$. Then we have

$$\begin{aligned} 2^{ls} & \left\| F^{-1} \varphi(t\,\cdot) \sum_{j=-\infty}^{l} \rho(2^{-j}\,\cdot) Ff \,|\, L_p \right\| \\ & \leqslant c 2^{l(s-s_1)} \left\| F^{-1} t^{-s_1} |x|^{-s_1} \varphi(t\,\cdot) h(t\,\cdot) \right. \\ & \quad \times \left. \sum_{j=-\infty}^{l} 2^{js_1} \tilde{\rho}(2^{-j}\,\cdot) Ff \,|\, L_p \right\| \\ & \leqslant c \sum_{j=-\infty}^{l} 2^{ls + s_1(j-l)} \| F^{-1} |tx|^{-s_1} \varphi(t\,\cdot) h(t\,\cdot) | L_1 \| \\ & \quad \times \| F^{-1} \tilde{\rho}(2^{-j}\,\cdot) Ff \,|\, L_p \|. \end{aligned} \tag{59}$$

The L_1-factor in (59) is independent of t and we obtain

$$\begin{aligned} \| F^{-1} |tx|^{-s_1} \varphi(t\,\cdot) h(t\,\cdot) | L_1 \| & = \| F^{-1} |x|^{-s_1} \varphi h | L_1 \| \\ & \leqslant c \| |x|^{-s_1} \varphi h | H_2^{\lambda} \| \end{aligned} \tag{60}$$

with $\lambda > n/2$ (cf. [21, (1.5.2/8)] or [18, lemma on p. 60]). By the assumptions for $\varphi(x)$ the right-hand side of (60) is finite. Then (59) yields

$$\begin{aligned} 2^{ls} & \left\| F^{-1} \varphi(t\,\cdot) \sum_{j=-\infty}^{l} \rho(2^{-j}\,\cdot) Ff \,|\, L_p \right\| \\ & \leqslant c \sum_{k=-\infty}^{0} 2^{k(s_1-s)}\, 2^{(l+k)s} \| F^{-1} \tilde{\rho}(2^{-k-l}\,\cdot) Ff \,|\, L_p \|. \end{aligned} \tag{61}$$

We raise (61) to the power q term by term. This can be done if one replaces $s_1 - s$ by a number δ with $s_1 - s > \delta > 0$. Then we integrate over t, where $2^{-l} \leqslant t \leqslant 2^{-l+1}$ and multiply with 2^l. Afterwards we take the sum over l and obtain that

$$\begin{aligned} & \left(\sum_{l=-\infty}^{\infty} 2^{lsq} \int_{2^{-l}}^{2^{-l+1}} \left\| F^{-1} \varphi(t\,\cdot) \sum_{j=-\infty}^{l} \rho(2^{-j}\,\cdot) Ff \,|\, L_p \right\|^q dt\, 2^l \right)^{1/q} \\ & \quad \leqslant c \| f \,|\, \dot{B}^s_{p,q} \|. \end{aligned} \tag{62}$$

Let $0 < p < 1$. Then (59) must be replaced by

$$2^{lsp} \left\| F^{-1}\varphi(t\,\cdot) \sum_{j=-\infty}^{l} \rho(2^{-j}\,\cdot)\, Ff \,|\, L_p \right\|^p$$
$$\leqslant c \sum_{j=-\infty}^{l} 2^{lsp+s_1(j-l)p}\, 2^{ln(1/p-1)p}$$
$$\times \|F^{-1}\,|tx|^{-s_1}\,\varphi(t\,\cdot)\, h(t\,\cdot)\,|\,L_p\|^p\, \|F^{-1}\tilde{\rho}(2^{-j}\,\cdot)\, Ff\,|\,L_p\|^p. \quad (63)$$

We used that

$$\|g_1 * g_2 \,|\, L_p\| \leqslant c b^{n(1/p-1)}\, \|g_1\,|\,L_p\|\, \|g_2\,|\,L_p\| \quad (64)$$

if $g_1 \in S'$, $g_2 \in S'$, $\operatorname{supp} Fg_1$ and $\operatorname{supp} Fg_2$ are contained in $\{y \,|\, |y| \leqslant b\}$ (cf. [21, (1.5.3/3)] or [18, p. 57]) (c is independent of b). Instead of (60) we have

$$2^{ln(1/p-1)p}\, \|F^{-1}\,|tx|^{-s_1}\,\varphi(t\,\cdot)\, h(t\,\cdot)\,|\,L_p\|^p$$
$$= 2^{ln((1/p-1)p} t^{-np} t^n\, \|F^{-1}\,|x|^{-s_1}\,\varphi h\,|\,L_p\|^p \leqslant c\, \|\,|x|^{-s_1}\,\varphi h\,|\,H_2^{\lambda}\|^p \quad (65)$$

with $\lambda > \sigma_p$ (cf. [21, (1.5.2/8)] or [18, lemma on p. 60]). If we put (65) in (63) we then obtain the counterpart of (61). The rest is the same as for $p \geqslant 1$. Consequently, we have (62) for all $0 < p \leqslant \infty$ and $0 < q \leqslant \infty$ (and $-\infty < s < \infty$). Now we deal with the terms with $j > l$. We modify the above function $h(x)$ by $h(x) \in S$ with $h(x) = 1$ if $\frac{1}{2} \leqslant |x| \leqslant 2$ and $h(x) = 0$ if either $|x| \geqslant 4$ or $|x| \leqslant \frac{1}{4}$. Let $0 < p < 1$ (the necessary modifications if $p \geqslant 1$ are clear now). Then (63) with $2^{-l} \leqslant t \leqslant 2^{-l+1}$, s_0 instead of s_1 and $\tilde{\rho}(x) = |x|^{s_0} \rho(x)$ (resp. (59) in the case $p \geqslant 1$) must be modified by

$$2^{lsp}\, \left\| F^{-1}\varphi(t\,\cdot) \sum_{j=l+1}^{\infty} \rho(2^{-j}\,\cdot)\, Ff \,|\, L_p \right\|^p$$
$$\leqslant c \sum_{j=l+1}^{\infty} 2^{lsp+s_0(j-l)p}\, 2^{jn(1/p-1)p}$$
$$\times \|F^{-1}\,|tx|^{-s_0}\,\varphi(t\,\cdot)\, h(2^{-j}\,\cdot)\,|\,L_p\|^p\, \|F^{-1}\tilde{\rho}(2^{-j}\,\cdot)\, Ff\,|\,L_p\|^p. \quad (66)$$

(65) must be replaced by

$$2^{jn(1/p-1)p}\, \|F^{-1}\,|tx|^{-s_0}\,\varphi(t\,\cdot)\, h(2^{-j}\,\cdot)\,|\,L_p\|^p$$
$$\leqslant \|F^{-1}\,|t\,2^j x|^{-s_0}\,\varphi(2^j t\,\cdot)\, h\,|\,L_p\|^p$$
$$\leqslant c\, \|\,|t\,2^j x|^{-s_0}\,\varphi(2^j t\,\cdot)\, h\,|\,H_2^{\lambda}\|^p \quad (67)$$

with $\lambda > \sigma_p$. We choose $\lambda = 1 + [\sigma_p]$. Then it follows from (24) and the properties of $h(x)$ that the right-hand side of (67) is finite and it can be estimated by a constant which is independent of j (and l). We put (67) in (66) and obtain that

$$2^{lsp} \left\| F^{-1}\varphi(t\,\cdot) \sum_{j=l+1}^{\infty} \rho(2^{-j}\,\cdot)\,Ff \,|\, L_p \right\|^p \leqslant c \sum_{k=1}^{\infty} 2^{(s_0-s)kp}\, 2^{(l+k)sp}\, \|F^{-1}\tilde{\rho}(2^{-k-l}\,\cdot)\,Ff \,|\, L_p\|^p. \tag{68}$$

Now we argue in the same way as after (61) and obtain the counterpart of (62) with $\sum_{j=l+1}^{\infty}$ instead of $\sum_{j=-\infty}^{l}$. This counterpart and (62) itself prove (56) (cf. (57)) (as has been mentioned, if $p \geqslant 1$ then the last calculations must be modified in the above way).

Step 2. We prove that there exists a constant c such that (under the hypotheses of Theorem 2)

$$\|f \,|\, \dot{B}^s_{p,q}\| \leqslant c \left(\int_0^{\infty} t^{-sq}\, \|F^{-1}\varphi(t\,\cdot)\,Ff \,|\, L_p\|^q\, \frac{dt}{t} \right)^{1/q} \tag{69}$$

holds for every $f \in \dot{B}^s_{p,q}$. Let $1 \leqslant p \leqslant \infty$. Then (49) yields

$$\|F^{-1}\rho(2^{-j}\,\cdot)\,Ff \,|\, L_p\| \leqslant c\, \|F^{-1}\varphi(2^{-j}\lambda\,\cdot)\,Ff \,|\, L_p\|, \tag{70}$$

where c is independent of j and λ with $1 \leqslant \lambda \leqslant 2$. Integration over λ, multiplication with 2^{js} and summation over the qth power yield (69). Let $0 < p < 1$. Let $\psi(x) \in S$ be a function with $\operatorname{supp}\psi \subset \{y \,|\, |y| \leqslant 2^{K+1}\}$ and $\psi(x) = 1$ if $|x| \leqslant 2^K$, where we choose the natural number K later on. Similar to (49) we have

$$|(F^{-1}\rho(2^{-l}\,\cdot)\,Ff)(x)| \leqslant \int_{R_n} \left| \left(F^{-1} \frac{\rho(2^{-l}\,\cdot)}{\varphi(2^{-l}\lambda\,\cdot)} \right)(y) (F^{-1}\varphi(2^{-l}\lambda\,\cdot)\,\psi(2^{-l}\,\cdot)\,Ff\Big)(x-y) \right| dy. \tag{71}$$

If $x \in R_n$ is fixed then the Fourier transform of the y-function in the integral in (71) has a support which is contained in a ball with the radius $c\, 2^{l+K}$, where c is independent of l and K. We can apply an inequality of Plancherel–Polya–Nikol'skij type (cf. [21, 1.3.2] or [18, p. 29]) and obtain that

$$|(F^{-1}\rho(2^{-l}\,\cdot)\,Ff)(x)|^p \leqslant c\, 2^{(l+K)pn(1/p-1)} \int_{R_n} \left| \left(F^{-1} \frac{\rho(2^{-l}\,\cdot)}{\varphi(2^{-l}\lambda\,\cdot)} \right)(y) \right|^p \times |(F^{-1}\varphi(2^{-l}\lambda\,\cdot)\,\psi(2^{-l}\,\cdot)\,Ff)(x-y)|^p\, dy. \tag{72}$$

Integration over x yields

$$\begin{aligned}
&\|F^{-1}\rho(2^{-l}\cdot)\,Ff\,|L_p\|^p \\
&\quad\leqslant c\,2^{Kn(1/p-1)p}\,\|F^{-1}\varphi(2^{-l}\lambda\cdot)\,\psi(2^{-l}\cdot)\,Ff\,|L_p\|^p \\
&\quad\leqslant c\,2^{Knp(1/p-1)}\,\|F^{-1}\varphi(2^{-l}\lambda\cdot)\,Ff\,|L_p\|^p \\
&\qquad+2^{Knp(1/p-1)}\,\|F^{-1}\varphi(2^{-l}\lambda\cdot)(1-\psi(2^{-l}\cdot))\,Ff\,|L_p\|^p.
\end{aligned} \tag{73}$$

The term on the right-hand side is just what we want (cf. (70)): We multiply with 2^{lsp}, integrate over λ with respect to $1\leqslant\lambda\leqslant 2$ and take the sum over the q/pth power with respect to l. That part which comes from the second term in (73) can be estimated in the same way as in (68), where $\varphi(t\cdot)$ is replaced by $\varphi(t\cdot)(1-\psi(2^{-l}\cdot))$ with $t=2^{-l}\lambda$. Then $\sum_{k=1}^{\infty}$ in (68) can be replaced by $\sum_{k=K}^{\infty}$. By assumption we have

$$(s-s_0)\,p>\tilde{\sigma}_p\,p=n(1/p-1)\,p. \tag{74}$$

We take the l_q-quasi-norm. Then (73) yields (after the above calculations)

$$\begin{aligned}
\|f\,|\dot{B}^s_{p,q}\|\leqslant C\left(\int_0^\infty t^{-sq}\,\|F^{-1}\varphi(t\cdot)\,Ff\,|L_p\|^q\,\frac{dt}{t}\right)^{1/q} \\
+c\,2^{Knp(1/p-1)}\,2^{-(s-s_0)pK}\,\|f\,|\dot{B}^s_{p,q}\|,
\end{aligned}$$

where c is independent of K. If we choose K big enough then we obtain the desired estimate from (74). The proof is complete.

4.3. *Proof of Theorem* 3 (*Outline*)

Step 1. We prove part (ii), the proof of part (i) is the same. Let $f\in F^s_{p,q}$ and let $\rho_0(x)$ be the function from 2.2. Then we have

$$\|f\,|F^s_{p,q}\|\sim\|F^{-1}\rho_0 Ff\,|L_p\|+\|F^{-1}(1-\rho_0)\,Ff\,|F^s_{p,q}\|. \tag{75}$$

We can apply (19) with $F^{-1}(1-\rho_0)\,Ff$ instead of f (similarly one can deal with (18) instead of (19)). Then we have

$$\begin{aligned}
&\|f\,|F^s_{p,q}\|\sim\|F^{-1}\rho_0 Ff\,|L_p\| \\
&\qquad+\left\|\left(\int_0^\infty t^{-sq}\,|(\varphi(t\cdot)\,F^{-1}(1-\rho_0)\,Ff)^*(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\|.
\end{aligned} \tag{76}$$

The crucial point of the proof is the following estimate,

$$\begin{aligned}
&\left\|\left(\int_0^\infty t^{-sq}\,|(\varphi(t\cdot)\,F^{-1}\rho_0 Ff)^*(\cdot)|^q\,\frac{dt}{t}\right)^{1/q}|L_p\right\| \\
&\qquad\leqslant c\,\|F^{-1}\rho_0 Ff\,|L_p\|,
\end{aligned} \tag{77}$$

where c is independent of f. Let us take it for granted that (77) is valid. Then (76) yields

$$\|f|F^s_{p,q}\| \sim \|F^{-1}\rho_0 Ff|L_p\| + \left\| \left(\int_0^\infty t^{-sq} |(\varphi(t\cdot)f)^*(\cdot)|^q \frac{dt}{t} \right)^{1/q} |L_p \right\|. \tag{78}$$

In [21, (2.5.9/37)] or [20, p. 1109] we proved that

$$\|F^{-1}\rho_0 Ff|L_p\| \leqslant c_\varepsilon \|f|L_p\| + \varepsilon \|f|F^s_{p,q}\|, \tag{79}$$

where $\varepsilon > 0$ is an arbitrary number [for this estimate one needs that $s > \max(0, n(1/p - 1))$]. However, (78) and (79) prove that (28) is an equivalent quasi-norm in $F^s_{p,q}$. We must prove (77). Let $g = F^{-1}\rho_0 Ff$. Then we have

$$(\varphi(t\cdot)g)^*(x) = \sup_{y \in R_n} (1 + |y|^a)^{-1} |(F^{-1}\varphi(t\cdot)Fg)(x - ty)| \tag{80}$$

(cf. (14)). By standard calculation it follows that

$$(F^{-1}\varphi(t\cdot)Fg)(x - ty) = \int_{R_n} (F^{-1}\varphi)(z)\, g(x - ty - tz)\, dz. \tag{81}$$

Let

$$g^*(x) = \sup_{u \in R_n} \frac{|g(x-u)|}{1 + |u|^a} \tag{82}$$

be the maximal function of the entire analytic function g. If $t > 1$, then (81) yields

$$|(F^{-1}\varphi(t\cdot)Fg)(x - ty)| \leqslant ct^a(1 + |y|^a) \int_{R_n} |(F^{-1}\varphi)(z)|\, (1 + |z|^a)\, dz \cdot g^*(x). \tag{83}$$

The assumptions for φ ensure that the integral in (83) converges. Consequently,

$$(\varphi(t\cdot)g)^*(x) \leqq ct^a g^*(x), \qquad t \geqslant 1. \tag{84}$$

If $0 < t \leqslant 1$, then we replace $\varphi(x)$ in (81) by $|x|^L \psi(x)$, which is reasonable (cf. (15)). We have

$$(F^{-1}\varphi)(z) = (F^{-1}|u|^L FF^{-1}\psi)(z)$$

and we shift $F^{-1}|u|^L F$ in (81) from ψ to g. Elementary calculations yield

$$(F^{-1}\varphi(t\,\cdot)\,Fg)(x-ty)$$
$$= t^L \int_{R_n} (F^{-1}\psi)(z)(F\,|u|^L\,F^{-1}g)(x-ty-tz)\,dz. \tag{85}$$

Our assumptions ensure that the integral in (83) with ψ instead of φ converges. Then the counterpart of (84) reads as

$$(\varphi(t\,\cdot)\,g)^*\,(x) \leqslant ct^L(F\,|u|^L\,F^{-1}g)^*\,(x), \qquad t \leqslant 1. \tag{86}$$

We choose a and L such that $s > a > n/\min(p, q)$ and $L > s$. Then we have

$$\left\| \left(\int_0^\infty t^{-sq}\,|(\varphi(t\,\cdot)\,g)^*\,(\cdot)|^q\,\frac{dt}{t} \right)^{1/q} |L_p \right\|$$
$$\leqslant c\,\|\,g^*|L_p\| + c\,\|(F\,|u|^L\,F^{-1}g)^*|L_p\|. \tag{87}$$

We recall that $g \in L_p$ and $\operatorname{supp} Fg \subset \{y\,|\,|y| \leqslant 2\}$. The theory of these L_p-spaces of entire analytic functions has been developed in [21, Chap. 1] and [18, Chap. 1]. In particular it follows that $|u|^L$ is a Fourier multiplier in these spaces and that the L_p-quasi-norms of the above maximal functions g^* and $(F\,|u|^L\,F^{-1}g)^*$ can be estimated from above by $c\,\|\,g|L_p\|$. Now (77) follows from (87) and we are through.

Step 2. The proof of (i) is the same as in Step 1. We prove part (iii). The counterpart of (76) is obvious. Instead of (77) we must prove that there exists a constant c such that

$$\left(\int_0^\infty t^{-sq}\,\|(F^{-1}\varphi(t\,\cdot)\,Fg)(\cdot)|L_p\|^q\,\frac{dt}{t} \right)^{1/q} \leqslant c\,\|\,g|L_p\| \tag{88}$$

with $g = F^{-1}\rho_0 Ff$ and $f \in B^s_{p,q}$. If (88) is established, then the counterparts of (78) and (79) yield the desired proof. Let $1 \leqslant p \leqslant \infty$ and $t \geqslant 1$. Then we use (81) with $y = 0$ and obtain

$$\|F^{-1}\varphi(t\,\cdot)\,Fg|L_p\| \leqslant c\,\|\,g|L_p\|, \qquad t \geqslant 1. \tag{89}$$

If $1 \leqslant p \leqslant \infty$ and $t \leqslant 1$ then we use (85) with $y = 0$ and $L = s_1 > s$ (cf. Theorem 2). We have

$$\|F^{-1}\varphi(t\,\cdot)\,Fg|L_p\| \leqslant ct^{s_1}\,\|F^{-1}\,|u|^{s_1}\,Fg|L_p\|, \qquad t \leqslant 1. \tag{90}$$

By the same arguments as above we obtain (88). Let $0 < p < 1$ and $t \geqslant 1$. Then (81) yields

$$|(F^{-1}\varphi(t\,\cdot)\,Fg)(x)|$$

$$\leqslant cg^*(x)^{1-p}\,t^{a(1-p)}\int_{R_n}|(F^{-1}\varphi)(z)|\,(1+|z|^a)^{1-p}\,|g(x-tz)|^p\,dz, \qquad (91)$$

where $a > n/p$ is sufficient for the application of the maximal inequatility for $g \in L_p$ (cf. [21, 1.4.1] or [18, p. 36]). In particular we can choose a in such a way that $s > a(1-p) > n(1/p-1)$. We raise (91) to the power p, integrate over $x \in R_n$ and apply Hölder's inequality with respect to $p + (1-p) = 1$. The result reads as

$$\|F^{-1}\varphi(t\,\cdot)\,Fg\,|L_p\| \leqslant ct^{a(1-p)}\,\|g^*\,|L_p\|^{1-p}\,\|g\,|L_p\|^p$$
$$\leqslant c' t^{a(1-p)}\,\|g\,|L_p\|, \qquad t \geqslant 1, \qquad (92)$$

which is the counterpart of (89). If $0 < t \leqslant 1$ (and $0 < p < 1$) then (86) with $L = s_1$ yields (90). This shows that (88) holds also if $0 < p < 1$. The proof is complete.

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Note added in proof. As far as characterizations of $\dot{F}^s_{p,q}$ with $1 < p < \infty$ and $1 < q < \infty$ are concerned (cf. Corollary 2 (ii)) we refer to [22, 23].

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